MINISTRY OF EDUCATION, SINGAPORE
in collaboration with
CAMBRIDGE ASSESSMENT INTERNATIONAL EDUCATION
General Certificate of Education Advanced Level
Higher 2

FURTHER MATHEMATICS 9649/02

Paper 2 **For examination from 2025**

SPECIMEN PAPER

3 hours

Additional Materials: Printed Answer Booklet
List of Formulae and Results (MF27)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, index number and name on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do **not** use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE ON ANY BARCODES.

Answer **all** questions.
Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are **not** allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.
You must show all necessary working clearly.

The number of marks is given in brackets [ ] at the end of each question or part question.

Singapore Examinations and Assessment Board

Cambridge Assessment International Education

Section A: Pure Mathematics [50 marks]

1 Let P be the set of polynomials with real coefficients of the form $a_0 + a_1x + a_2x^2$.

(a) **(i)** Show that P is closed under the usual operations of addition, and multiplication by a scalar. [2]

(ii) Assuming that P is a vector space over the field of real numbers, write down the standard basis for this vector space. [1]

Every polynomial with real coefficients of the form $a_0 + a_1x + a_2x^2$ can be represented by the 3×1 column vector $\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$.

The D-transformation, which acts on elements of P, represents differentiation with respect to x. So, for example, the statement $D(1 + 2x + 3x^2) = 2 + 6x$ can be expressed as

$$D\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix}$$

(b) **(i)** Show that D may be regarded as a linear transformation from P to P. [2]

(ii) Write down the matrix representing D and find the null space of D. [2]

(iii) Describe what is represented by the transformation D^2. [1]

2 For $xy \neq 1$, a surface S is given by the equation $z = \dfrac{x+y}{xy-1}$.

(a) Prove that S has no maximum or minimum turning points. [5]

(b) Let P be a plane with equation of the form $x + y + z = k$, where k is a constant, and let (a, b, c) be a point that lies both on P and on S. Show that $abc = k$ also. [2]

3 The space-transformation T is given by the matrix $\mathbf{A} = \begin{pmatrix} 3 & -2 & -1 \\ 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$.

(a) Write down the characteristic equation of $\mathbf{A}$ and hence state its three eigenvalues. [2]

(b) **(i)** For each eigenvalue, give the full set of associated eigenvectors. [5]

(ii) Give a geometrical interpretation of each eigenvalue's set of eigenvectors in relation to T. [2]

4 Let $f(x) = \sin^{-1}(2x) - \sin^{-1}(x)$ for $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$.

(a) Use Simpson's rule with five ordinates to find an approximation to the area below the curve $y = f(x)$ and above the x-axis between $x = 0$ and $x = \frac{1}{2}$. Give your answer to 4 decimal places. [3]

(b) **(i)** Use a 'sign-change' method to show that the equation $f(x) = \frac{1}{2}$ has a root α in the interval $(0, \frac{1}{2})$. [1]

A student writes a computer program to find the value of α to 4 decimal places. He first writes the equation $f(x) = \frac{1}{2}$ in the form $x = F(x)$ and then runs the program to generate a sequence of approximations $\{x_n\}$ using the recurrence relation

$$x_0 = 0.5 \text{ and } x_n = F(x_{n-1}) \text{ for } n \geqslant 1.$$

At each intermediate stage of the process, the computer gives the value of x_n, correct to 4 decimal places, as the output, while still retaining a more accurate value for further calculations.

(ii) In his first attempt, the student uses $F(x) = \sin(\sin^{-1}(2x) - \frac{1}{2})$.

Calculate the computer's output for x_1 and explain why the programme cannot calculate x_2. [2]

(iii) In his second attempt, the student uses $F(x) = \frac{1}{2}\sin(\sin^{-1}x + \frac{1}{2})$ and believes he has obtained the value of α correctly to 4 decimal places when two successive output values are equal.

A Calculate the computer's output for $x_1, x_2, \dots, x_7$. [1]

B Explain why the student's programme fails to find the correct value of α to the required degree of accuracy. [1]

C By continuing this second process further, find the value of α to 4 decimal places, and demonstrate how the student could verify its correctness. [2]

5 The population of fish in a lake is P (in thousands) and is assumed to be a function of time t (in years). The growth of this population of fish is governed by the logistic equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{b}\right) - H(P).$$

The constants b and k are positive, and $H(P)$ is a harvesting term.

(a) **(i)** Suppose first that $H(P)$ is a constant, h.

Show how the number of equilibrium population values of P depends on the nature of the term $h - \frac{1}{4}bk$. [3]

(ii) Sketch a graph of P against t in the case when $h = bk$, justifying the behaviour of P. [2]

A second population model, also with harvesting term, is given by $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{b}\right)\left(1 - \frac{a}{P}\right)$.

In this case, k, a and b are positive constants and $a < b$. The initial population is given by $P(0) = p$, where $p > 0$.

(b) **(i)** Show that, in this case, the harvesting term, $H(P)$, is $ka\left(1 - \frac{P}{b}\right)$. [1]

(ii) State the significance of each of the constants a and b and sketch a graph of P against time t, to show the behaviour of the fish population for each of the different cases that arise according to the various possible values of p. [5]

(iii) Given that $k = 12.5$, $a = 20$ and $b = 100$, use one iteration of Euler's Method to estimate the value of p, to the nearest whole number, for which the population dies out after 1 year. [5]

Section B: Probability and Statistics [50 marks]

6 In an opinion poll, a random sample of 1000 voters were asked which political party they intended to vote for. Out of the 1000 voters, 469 voters said that they intended to vote for Party X.

(a) Calculate an approximate 95% confidence interval for the proportion of voters in the population who intend to vote for Party X. [2]

A simple formula used to calculate the end points an approximate 95% confidence interval for p is

$$p = \hat{p} \pm \frac{1}{\sqrt{n}},$$

where $\hat{p}$ is the sample proportion and n is the sample size.

(b) Calculate an approximate confidence interval using the simple formula. [1]

(c) **(i)** Show that the simple formula gives an approximate interval that will always be wider than the interval given by the standard formula. [3]

(ii) What does this indicate about the confidence level of the approximate interval given by the simple formula? [1]

7 A random sample of adults, aged 60 or over, were asked about their state of health and their level of physical activity. Their responses are summarised in the table.

		Level of physical activity		
		Low	Medium	High
State of health	Good	13	26	14
	Medium	55	119	69
	Poor	29	27	13

(a) Carry out a chi-squared test for the independence of the two factors, level of physical activity and state of health, for adults aged 60 or over. [7]

(b) Discuss what the test indicates about the association, if any, between the two factors. You should refer to the p value for your test and to any combination(s) of state of health and level of physical activity which may be of particular concern. [2]

8 The outer ring-road around a large city is approximately circular and of radius 20 km. In a simulated emergency situation it is supposed that two major accidents occur simultaneously at random points on the ring-road. An emergency helicopter has to fly between the two accident locations, and it is required to calculate the expected distance the helicopter will have to travel. The situation is modelled as follows.

Two points A and B are independently and randomly chosen to lie on the circumference of a circle with centre O and radius 1. The non-reflex angle AOB is uniformly distributed on the interval from 0 to π. The chord AB has length denoted by the random variable X.

(a) **(i)** By considering $P(X < x)$ or otherwise, show that the probability density function of X is $f(x)$, where

$$f(x) = \begin{cases} \dfrac{2}{\pi\sqrt{4-x^2}} & 0 \leqslant x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

[5]

(ii) Sketch this probability density function. [2]

(b) Determine the exact value of $E(X)$. [2]

(c) State the expected distance the helicopter will have to fly. [1]

9 A food manufacturer wished to investigate whether adding a preservative to a bottled sauce affected the taste quality. A tasting panel, consisting of twelve randomly chosen potential purchasers, was used. Each person tasted the sauce with and without the preservative and gave a taste quality score from 0 to 20 to each.

The taste quality scores given by the twelve members of the panel were as follows.

Taster	A	B	C	D	E	F	G	H	I	J	K	L
With preservative	15	17	5	13	11	7	9	7	15	18	5	3
Without preservative	14	19	8	17	16	13	17	16	5	7	17	17

(a) State what modelling assumption would be needed for a paired-sample t test to be used in this context. [1]

(b) The manufacturer would like to use a chi-squared goodness-of-fit test to investigate whether the modelling assumption in part **(a)** is valid. By considering degrees of freedom and expected frequencies, explain why it is not possible to carry out such a test with a sample of fewer than 20 people. [3]

(c) Carry out a Wilcoxon matched pair signed rank test, at the 5% level of significance, to investigate whether adding the preservative affects the taste quality. [6]

(d) It was subsequently found that the two scores for **one** of the tasters had been recorded the wrong way round. The result of correcting the error was that the conclusion of the test was changed. Determine which of the tasters could have had their scores recorded the wrong way round. [2]

10 The random variable X has a Poisson distribution with parameter λ. The random variable Y is defined as follows.

$$P(Y=r)=\begin{cases} kP(X=r) & r=1,2,3,\ldots \\ 0 & \text{otherwise,} \end{cases}$$

where k is a constant.

(a) **(i)** Find the exact value of k. [2]

(ii) Hence determine the value of $E(Y)$. [2]

(b) Use the fact that $E(X^2)=\lambda^2+\lambda$ to determine $E(Y^2)$. [2]

The random variable Y is now used as a model of the number of e-mails received by a secretary in a fixed interval of time, t minutes.

(c) The mean of 100 observations of Y is denoted by m.

Show that a suitable estimate of λ is given by the equation

$$\frac{\lambda}{1-e^{-\lambda}}=m. \qquad (*)$$ [2]

(d) Give an approximate form of equation (*) when t is large. [1]

(e) Show that, when t is small, $\lambda \approx 2\left(1-\frac{1}{m}\right)$. [3]